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LIFTING AEROFOIL CALCULATION USING THE BOUNDARY ELEMENT METHOD

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SUMMARY

The boundary integral formulation and boundary element method are extended to include lifting flow problems. This involves inclusion of a branch cut in the flow field and imposition of a Kutta condition to determine the circulation, Γ. Additional boundary integral contributions arise from the cut surface. Techniques for calculating Γ are developed and we treat, in particular, a superposition procedure which permits very efficient computation. Numerical results are presented for an NACA0012 aerofoil at several angles of attack.

KEY WORDS Boundary Elements Lifting Aerofoil Potential Flow

INTRODUCTION

Integral equation techniques have been extensively used in computing incompressible potential flows, principally by means of panel methods. These methods are based upon ideas of superposition of potentials due to, for example, source and doublet distributions.[1] A different type of boundary integral equation can be obtained by applying an appropriate Green's formula to recast the linear potential equation in the interior as an equation involving integrals on the boundary.[2] A finite element expansion for the solution field may then be introduced on a discretization of the boundary and an approximate solution of the boundary integral equation thereby obtained. This procedure has been termed the boundary element method.[3,4] The ideas have been developed for several classes of problems and studied in particular for linear potential flows governed by Laplace's equation.[5,6] Many of the applications of the boundary element method have concerned porous flow problems and similar problems in water resources.[7,8] Studies of potential aerodynamic flows with boundary elements have been limited to non-lifting profiles and, as such, represent simply an application of previously well-developed ideas.[9]

The boundary element method and panel method have some strong similarities, principally due to their use of boundary integral relations and the classical ideas of potential theory. There are, however, some major conceptual distinctions; the panel method is based upon superposition using for instance sources, doublets or vortices with solution determined for example from the discrete satisfaction of boundary flux conditions; in the boundary element method we use a finite element expansion and a discrete approximation of the boundary integral equation.

Standard finite element variational methods have been previously developed and applied to lifting aerofoil problems.[10,11] Our purpose here is to extend the boundary element method to treat these lifting flows. To achieve this, we shall introduce a 'branch cut' to develop both

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a direct integral formulation in which circulation Γ is obtained iteratively and also a superposition formulation. The latter approach leads to a particularly efficient algorithm which has been applied in numerical studies of an NACA0012 aerofoil at angle of attack.

METHOD

1. Flow problem

We shall limit the treatment here to incompressible two-dimensional flows governed by Laplace's equation†

$$\Delta\phi = 0 \tag{1}$$

where ϕ is the potential for flow past an aerofoil in the flow domain Ω. The associated boundary conditions are that there be no flow through the aerofoil boundary $\partial\Omega_a$,

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega_a \tag{2}$$

and that in the far field

$$\nabla\phi \to \mathbf{U}_\infty \quad \text{as} \quad r^2 = x^2 + y^2 \to \infty \tag{3}$$

where $\mathbf{U}_\infty$ is the specified uniform flow at infinity.

On a remote far-field boundary $\partial\Omega_f$ we can approximate (3) by

$$\frac{\partial\phi}{\partial n} = \mathbf{U}_\infty \cdot \mathbf{n} \quad \text{on } \partial\Omega_f \tag{4}$$

where $\mathbf{n}$ is the unit outward normal from $\partial\Omega_f$.

Finally, the unknown circulation Γ is to be determined from the Kutta condition which asserts that the flow should leave smoothly from the trailing edge. In turn, this is equivalent to the requirement that the trailing edge be a stagnation point in the flow.[10]

$$\mathbf{u} = \mathbf{0} \quad \text{at the trailing edge } (x_E, y_E) \tag{5}$$

If we consider any simple closed contour $\mathscr{C}$ enclosing the aerofoil, then the circulation Γ is defined by

$$\Gamma = \oint_{\mathscr{C}} \mathbf{u} \cdot d\mathbf{s} = \oint_{\mathscr{C}} \nabla\phi \cdot d\mathbf{s} = \phi(P) - \phi(Q) \tag{6}$$

where $\mathbf{u}$ is the velocity and P, Q are a pair of adjacent points on either side of a branch cut from the aerofoil (Figure 1). The branch cut consists of the pair of slit surfaces shown and designated $\partial\Omega_{s^+}$ and $\partial\Omega_{s^-}$ for upper and lower surfaces, respectively.

Hence, across the slit surface we have

$$[\![\phi]\!] = \Gamma \quad \text{on } \partial\Omega_s \tag{7}$$

and the velocity is continuous, so

$$\left[\!\!\left[\frac{\partial\phi}{\partial n}\right]\!\!\right] = 0 \quad \text{on } \partial\Omega_s \tag{8}$$

where $[\![\cdot]\!]$ denotes the jump across the branch cut. Both relations (7) and (8) will be particularly important in our boundary integral formulation for the lifting aerofoil problem.

† In continuing studies we have recently extended the approach to subcritical compressible flows and the method is currently being implemented and tested.

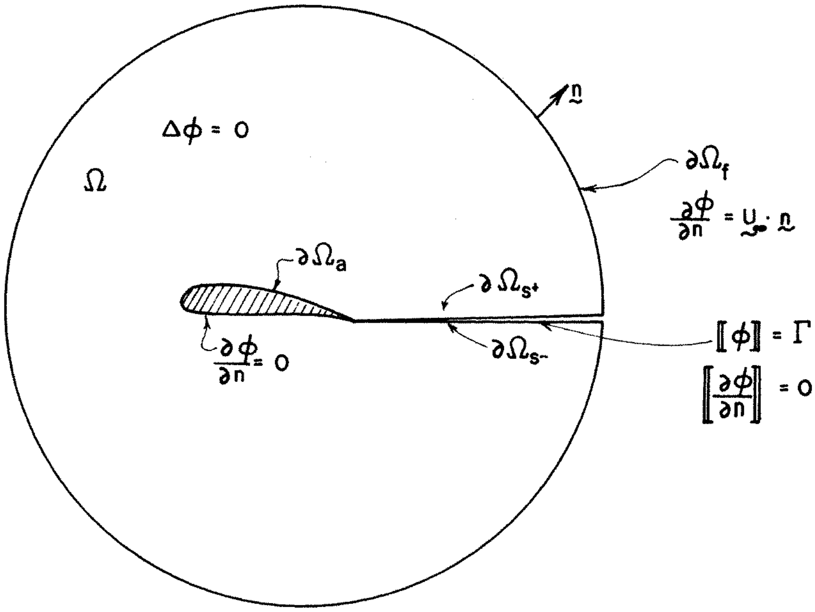

Figure 1. Flow domain Ω showing aerofoil, branch cut and far-field boundaries $\partial\Omega_a$, $\partial\Omega_{s^\pm}$, $\partial\Omega_f$ and associated boundary conditions

2. *Boundary integral equation*

Let $\partial\Omega$ be the total boundary comprised of aerofoil, far-field boundary and both slit surfaces: $\partial\Omega = \partial\Omega_a \cup \partial\Omega_f \cup \partial\Omega_{s^+} \cup \partial\Omega_{s^-}$. The boundary integral equation is derived from Green's third identity for the Laplacian

$$\int_\Omega (\phi\,\Delta\chi - \chi\,\Delta\phi)\,dx\,dy = \int_{\partial\Omega} \left(\phi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\phi}{\partial n}\right) ds \tag{9}$$

Select χ as the fundamental solution for the Laplacian

$$\Delta\chi = \delta(\mathbf{x} - \boldsymbol{\zeta}) \tag{10}$$

In two dimensions χ is the familiar potential solution for a point source

$$\chi(\mathbf{x}; \boldsymbol{\zeta}) = \frac{1}{2\pi}\ln r \tag{11}$$

where $r = |\mathbf{x} - \boldsymbol{\zeta}|$ and $\boldsymbol{\zeta}$ is a point in the interior of Ω.

Using (1) and (10) in (9) and simplifying, we obtain[4]

$$\beta(\mathbf{x})\phi(\mathbf{x}) = \int_{\partial\Omega} \left\{\phi(s)\frac{\partial\chi}{\partial n}(\mathbf{x}; s) - \chi(\mathbf{x}; s)\frac{\partial\phi}{\partial n}(s)\right\} ds \tag{12}$$

Where we have denoted $\phi(\mathbf{x}(s))$ as $\phi(s)$, etc. In deriving (12) we have employed the symmetry of χ to interchange $\mathbf{x}$ and $\boldsymbol{\zeta}$ for convenience and have then used the Dirac delta property to write

$$\int_\Omega \delta(\mathbf{x} - \boldsymbol{\zeta})\phi(\boldsymbol{\zeta})\,d\boldsymbol{\zeta} = \begin{cases} \beta(\mathbf{x})\phi(\mathbf{x}) & \text{if } \mathbf{x} \in \bar{\Omega} \\ 0 & \text{if } \mathbf{x} \notin \bar{\Omega} \end{cases} \tag{13}$$

where

$$\beta(\mathbf{x}) = \theta/2\pi \tag{14}$$

and θ is the interior angle at $\mathbf{x}$. (If $\mathbf{x}$ is in the interior Ω of the flow field, then $\theta = 2\pi$ and $\beta = 1$; if $\mathbf{x}$ is on the boundary $\partial\Omega$ where there is a continuously turning tangent, then $\theta = \pi$ and $\beta = 1/2$, etc.

Let g denote the integrand on the right in (12) so that here we have for a point $\mathbf{x}$ on $\partial\Omega$, and the lifting aerofoil,

$$\begin{aligned}\beta(\mathbf{x})\phi(\mathbf{x}) &= \oint_{\partial\Omega_a} g\,\mathrm{d}s + \oint_{\partial\Omega_f} g\,\mathrm{d}s + \int_{\partial\Omega_{s^+}} g\,\mathrm{d}s + \int_{\partial\Omega_{s^-}} g\,\mathrm{d}s \\ &= I_a + I_f + I_{s^+} + I_{s^-}\end{aligned} \tag{15}$$

Considering each of the integrals in (15), we have: on the aerofoil, since $\partial\phi/\partial n = 0$ by (2),

$$I_a = \oint_{\partial\Omega_a} \left(\phi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\phi}{\partial n}\right)\mathrm{d}s = \oint_{\partial\Omega_a} \phi\frac{\partial\chi}{\partial n}\mathrm{d}s \tag{16}$$

On the far-field boundary, since $\partial\phi/\partial n = \mathbf{U}_\infty \cdot \mathbf{n}$,

$$I_f = \oint_{\partial\Omega_f} \left(\phi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\phi}{\partial n}\right)\mathrm{d}s = \oint_{\partial\Omega_f} \left(\phi\frac{\partial\chi}{\partial n} - \chi\mathbf{U}_\infty \cdot \mathbf{n}\right)\mathrm{d}s \tag{16}$$

Finally, combining the integrals on the cut:

$$\begin{aligned}I_{s^+} + I_{s^-} &= \int_{\partial\Omega_{s^+}} \left(\phi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\phi}{\partial n}\right)\mathrm{d}s + \int_{\partial\Omega_{s^-}} \left(\phi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\phi}{\partial n}\right)\mathrm{d}s \\ &= \int_{\partial\Omega_{s^+}} \left([\![\phi]\!]\frac{\partial\chi}{\partial n} - \chi\left[\!\!\left[\frac{\partial\phi}{\partial n}\right]\!\!\right]\right)\mathrm{d}s \\ &= \int_{\partial\Omega_{s^+}} [\![\phi]\!]\frac{\partial\chi}{\partial n}\mathrm{d}s \\ &= \Gamma\int_{\partial\Omega_{s^+}} \frac{\partial\chi}{\partial n}\mathrm{d}s\end{aligned} \tag{18}$$

where we have used the direction of integration on $\partial\Omega_{s^-}$ and relations (7) and (8) to obtain the result in (18).

Combining (16)–(18) in the boundary integral equation (15),

$$\beta(\mathbf{x})\phi(\mathbf{x}) = \int_{\partial\Omega_a} \phi\frac{\partial\chi}{\partial n}\mathrm{d}s + \int_{\partial\Omega_f} \left(\phi\frac{\partial\chi}{\partial n} - \chi\mathbf{U}_\infty \cdot \mathbf{n}\right)\mathrm{d}s + \Gamma\int_{\partial\Omega_s} \frac{\partial\chi}{\partial n}\mathrm{d}s \tag{19}$$

Since χ is known we can transpose terms to rewrite (19) as

$$\beta(\mathbf{x})\phi(\mathbf{x}) - \int_{\partial\Omega_a} \phi\frac{\partial\chi}{\partial n}\mathrm{d}s - \int_{\partial\Omega_f} \phi\frac{\partial\chi}{\partial n}\mathrm{d}s = -\int_{\partial\Omega_f} \chi\mathbf{U}_\infty \cdot \mathbf{n}\,\mathrm{d}s + \Gamma\int_{\partial\Omega_s} \frac{\partial\chi}{\partial n}\mathrm{d}s \tag{20}$$

If in addition we use a far-field asymptotic approximation ϕ_f for ϕ on $\partial\Omega_f$ the integral equation simplifies further to

$$\beta(\mathbf{x})\phi(\mathbf{x}) - \int_{\partial\Omega_a} \phi\frac{\partial\chi}{\partial n}\mathrm{d}s = \int_{\partial\Omega_f} \left(\phi_f\frac{\partial\chi}{\partial n} - \chi\mathbf{U}_\infty \cdot \mathbf{n}\right)\mathrm{d}s + \Gamma\int_{\partial\Omega_s^+} \frac{\partial\chi}{\partial n}\mathrm{d}s \tag{21}$$

so that the unknown solution ϕ enters only for $\mathbf{x}$ on the aerofoil $\partial\Omega_a$.

The circulation Γ enters as an unknown in the integral equation. We can seek to find Γ iteratively by initially setting $\Gamma = 0$, solving, and then using the Kutta condition to adjust Γ. That is, the initial guess $\Gamma^{(0)}$ is given and we solve for $\phi^{(1)}$ and use (5) to correct $\Gamma^{(0)}$ to $\Gamma^{(1)}$, and so on. Subsequently in Section 4 we describe an alternative superposition method for linear lifting potential flows. The iterative method indicated here has also been used in standard finite element methods and in the extension of the present boundary element method to compressible lifting flows.

3. *Boundary element method*

Let us consider the formulation (21). Discretize $\partial\Omega_a$ to n elements $\partial\Omega_e$ each having N_e nodal degrees of freedom. At the trailing edge we require a pair of points: P for the upper surface element and Q for the lower surface element. At this pair-point we have

$$\phi_+ - \phi_- = \Gamma \tag{22}$$

A similar relation holds for the corresponding pair of points at the branch cut on $\partial\Omega_f$.

We shall consider isoparametric elements so that the number of degrees of freedom N_e describing the element shape (linear, quadratic, etc.) agree with the number defining the local approximation of ϕ on $\partial\Omega_a^e$. The finite element approximation has the form

$$\phi(\mathbf{x}) = \sum_{j=1}^{N} \phi_j \psi_j(\mathbf{x}) \tag{23}$$

Substituting in (20), we obtain

$$\beta(\mathbf{x}) \sum_{j=1}^{N} \phi_j \psi_j(\mathbf{x}) - \int_{\partial\Omega_a} \sum_{j=1}^{N} \phi_j \psi_j(\mathbf{x}) \frac{\partial \chi}{\partial n} \mathrm{d}s = \int_{\partial\Omega_f} \left(\phi_f \frac{\partial \chi}{\partial n} + \chi \mathbf{U}_\infty \cdot \mathbf{n} \right) \mathrm{d}s + \Gamma \int_{\partial\Omega_{s^+}} \frac{\partial \chi}{\partial n} \mathrm{d}s \tag{24}$$

Now consider the collocation of (24) at points $\mathbf{x} = \mathbf{x}_i$ on $\partial\Omega_a$. Note that the solution values and normal derivative on the cut, although unknown, do not enter the boundary element approximations except in the form of the circulation term in (24). There are nN_e nodal values of ϕ on the aerofoil. We have one relation (22) and can obtain the remaining equations defining the boundary element system by collocating (24) at points in the boundary elements. For example, with linear elements we may collocate at the element nodes.

4. *Superposition (split) approach*

We now consider a different formulation in which the original solution is split as the superposition of two potentials satisfying slightly different problems derived from (1)–(8).[10,11] This leads us to develop a boundary element method that is particularly efficient and which is implemented in subsequent numerical studies.

Let us express the potential solution ϕ as a superposition of potentials v and w, in the form

$$\phi = \Gamma v + w \tag{25}$$

where Γ is the unknown circulation. It follows from the previous discussion (see Figure 1) that

$$[\![\phi]\!] = \Gamma[\![v]\!] + [\![w]\!] = \Gamma \quad \text{across } \partial\Omega_s \tag{26}$$

whence we can set

$$[\![v]\!] = 1 \quad \text{and } [\![w]\!] = 0 \quad \text{across } \partial\Omega_s \tag{27}$$

in the subsidiary problems. Similarly, $[\![\partial\phi/\partial n]\!]=0$ on $\partial\Omega_s$ implies

$$[\![\partial v/\partial n]\!]=[\![\partial w/\partial n]\!]=0 \quad \text{on } \partial\Omega_s \tag{28}$$

From the potential flow equation $\Delta\phi=0$, we have

$$\Delta v=0 \quad \text{and} \quad \Delta w=0 \quad \text{in } \Omega \tag{29}$$

Finally, the boundary conditions on $\partial\Omega_a$ and $\partial\Omega_f$ imply

$$\frac{\partial v}{\partial n}=\frac{\partial w}{\partial n}=0 \quad \text{on } \partial\Omega_a \tag{30}$$

and

$$\frac{\partial v}{\partial n}=0, \qquad \frac{\partial w}{\partial n}=\mathbf{U}_\infty\cdot\mathbf{n} \quad \text{on } \partial\Omega_f \tag{31}$$

The boundary-value problems for v and w are, respectively,

$$\left.\begin{array}{l} \Delta v=0 \quad \text{in } \Omega \\ \dfrac{\partial v}{\partial n}=0 \quad \text{on } \partial\Omega_a \quad \text{and} \quad \partial\Omega_f \\ [\![v]\!]=1, \qquad [\![\partial v/\partial n]\!]=0 \quad \text{on } \partial\Omega_s \end{array}\right\} \tag{32}$$

and

$$\left.\begin{array}{l} \Delta w=0 \quad \text{in } \Omega \\ \dfrac{\partial w}{\partial n}=0 \quad \text{on } \partial\Omega_a, \qquad \dfrac{\partial w}{\partial n}=\mathbf{U}_\infty\cdot\mathbf{n} \quad \text{on } \partial\Omega_f \\ [\![w]\!]=0, \qquad [\![\partial w/\partial n]\!]=0 \quad \text{on } \partial\Omega_s \end{array}\right\} \tag{33}$$

The boundary integral equation for v in (32) is obtained following the earlier procedure (see (20)) to be

$$\beta(\mathbf{x})v(\mathbf{x})-\int_{\partial\Omega_a} v\frac{\partial\chi}{\partial n}\,\mathrm{d}s-\int_{\partial\Omega_f} v\frac{\partial\chi}{\partial n}\,\mathrm{d}s=\int_{\partial\Omega_{s+}}\frac{\partial\chi}{\partial n}\,\mathrm{d}s \tag{34}$$

and, similarly, for w in (33),

$$\beta(\mathbf{x})w(\mathbf{x})-\int_{\partial\Omega_a} w\frac{\partial\chi}{\partial n}\,\mathrm{d}s-\int_{\partial\Omega_f} w\frac{\partial\chi}{\partial n}\,\mathrm{d}s=-\int_{\partial\Omega_f}\chi\mathbf{U}_\infty\cdot\mathbf{n}\,\mathrm{d}s \tag{35}$$

If we introduce the asymptotic far-field approximation ϕ_f for ϕ (as in (21)), we obtain asymptotic boundary data on $\partial\Omega_f$ for v_f and w_f from (25) in the form of a uniform flow and uniform circulation. (Recall that the complex velocity for uniform flow past a unit cylinder with circulation Γ is

$$F(z)=|\mathbf{U}_\infty|\left(z+\frac{1}{z}\right)-\frac{\Gamma}{2\pi i}\log z, \qquad z=x+iy \tag{36}$$

and can be used to define the asymptotic far-field approximation). In this case we can introduce v_f and w_f on $\partial\Omega_f$ into (34) and (35) to obtain analogous integral equations to that in (21).

The boundary element equations are formed as before. A discretization of the aerofoil boundary $\partial\Omega_a$ is introduced ($\partial\Omega_a$ and $\partial\Omega_f$ in (34)–(35) are used with v and w on $\partial\Omega_f$

unspecified). The finite element expansions for v and w on an element $\partial\Omega^e$ are defined by

$$v_e = \sum_{j=1}^{N_e} v_j\psi_j^e(\mathbf{x}), \qquad w_e = \sum_{j=1}^{N_e} w_j\psi_j^e(\mathbf{x}) \tag{37}$$

where ψ_j^e are obtained from local element basis functions $\{\hat{\psi}_i\}$ for $i = 1, 2, \ldots, N_e$ defined on the master element $\partial\hat{\Omega} = [-1, 1]$, and where $\partial\Omega_e$ is defined by an isoparametric map of the form

$$x = \sum_{j=1}^{N_e} x_j\hat{\psi}_j(\zeta), \qquad y = \sum_{j=1}^{N_e} y_j\hat{\psi}_j(\zeta), \qquad \zeta \in [-1, 1]. \tag{38}$$

and N_e is the number of degrees of freedom (nodes) for the element.

Substituting (37) in (34) and (35) and using the asymptotic far-field approximations v_f and w_f, we obtain

$$\beta(\mathbf{x})v(\mathbf{x}) - \sum_{e=1}^{E} \int_{\partial\Omega^e} v_e \frac{\partial\chi}{\partial n} \, ds = \int_{\partial\Omega_f} v_f \frac{\partial\chi}{\partial n} \, ds + \int_{\partial\Omega_{s+}} \frac{\partial\chi}{\partial n} \, ds \tag{39}$$

and

$$\beta(\mathbf{x})w(\mathbf{x}) - \sum_{e=1}^{E} \int_{\partial\Omega^e} w_e \frac{\partial\chi}{\partial n} \, ds = \int_{\partial\Omega_f} \left(w\frac{\partial\chi}{\partial n} - \chi\mathbf{U}_\infty \cdot \mathbf{n} \right) ds \tag{40}$$

Collocating (39) at points $\{\mathbf{x}_i\}$, $i = 1, 2, \ldots, N$

$$\beta(\mathbf{x}_i)v(\mathbf{x}_i) - \sum_{e=1}^{E} \int_{\partial\Omega^e} \sum_{j=1}^{N_e} v_j\psi_j(s) \frac{\partial\chi}{\partial n}(\mathbf{x}_i; s) \, ds = \int_{\partial\Omega_f} v_f(s) \frac{\partial\chi}{\partial n}(\mathbf{x}_i; s) \, ds + \int_{\partial\Omega_{s+}} \frac{\partial\chi}{\partial n}(\mathbf{x}_i; s) \, ds \tag{41}$$

This, together with the condition $[\![v]\!] = 1$ at the trailing edge pair of points, yields the linear algebraic boundary element system for $\mathbf{v}$.

A similar system is obtained from (40)

$$\beta(\mathbf{x}_i)w(\mathbf{x}_i) - \sum_{e=1}^{E} \int_{\partial\Omega_e} \sum_{j=1}^{N_e} w_j\psi_j(s) \frac{\partial\chi}{\partial n}(\mathbf{x}_i; s) \, ds = \int_{\partial\Omega_f} w_f \frac{\partial\chi}{\partial n} \, ds - \int_{\partial\Omega_f} \chi(\mathbf{x}_i; s)\mathbf{U}_\infty \cdot \mathbf{n} \, ds \tag{42}$$

and system solution determines $\mathbf{w}$.

Since we have used the same discretization and element basis for v and w the coefficient matrices in the two boundary element systems are identical. This implies that we may decompose the coefficient matrix $\mathbf{A}$ in (41) to the product $\mathbf{LU}$ where $\mathbf{L}$ and $\mathbf{U}$ are the usual lower and upper triangular matrices of $\mathbf{LU}$ decomposition in Gaussian elimination; $\mathbf{L}$ and $\mathbf{U}$ are stored, and the system solution for both $\mathbf{v}$ and $\mathbf{w}$ reduces to inexpensive forward and backward substitution sweeps on their respective right hand sides.

The final step is the determination of Γ to reconstruct ϕ in (25). From (25) we have $\phi = \Gamma v + w$, so that the velocity components are

$$u_1 = \phi_x = \Gamma v_x + w_x, \qquad u_2 = \phi_y = \Gamma v_y + w_y \tag{43}$$

where $\phi_x = \partial\phi/\partial x$, etc.

Imposing the Kutta condition by requiring that the trailing edge be a stagnation point of the flow, we get on examining for instance the x-component of velocity

$$\Gamma = \frac{-w_x}{v_x} \quad \text{at the trailing edge } (x, y)_{\mathrm{TE}} \tag{44}$$

Hence the circulation Γ may be calculated by (44) from the computed boundary element solutions for v and w and these results determine ϕ in (25). An accurate estimate of Γ from (44) will be obtained if the derivatives w_x and v_x have been determined accurately at the trailing edge. However, accurate results at this point are computationally difficult to achieve and would require a very fine graded mesh in this region.

We can determine Γ alternatively by requiring that the flow leave smoothly from the trailing edge. Let 1, 2 represent the pair-point at the trailing edge with 1 corresponding to the upper surface element and 2 the lower surface element. From the potentials v and w we calculate the tangential velocities at 1 and 2 and denote them by V_1, V_2 and W_1, W_2. Then, for a smooth flow from the trailing edge, the upper surface velocity must equal the lower so that

$$\Gamma V_1 + W_1 = \Gamma V_2 + W_2 \tag{45}$$

whence

$$\Gamma = \frac{(W_2 - W_1)}{(V_1 - V_2)} \tag{46}$$

which can also be deduced from the condition above that the trailing edge be a stagnation point of the flow. The section lift coefficient $\mathscr{C}_l$ can be shown to be

$$\mathscr{C}_l = 2\Gamma \tag{47}$$

5. Several angles of attack

In the subsequent numerical studies we shall compute the flow solution for several angles of attack α. The previous superposition approach can be modified slightly to yield a scheme for computing these flow solutions very efficiently. Recall the form of the split boundary-value problems for $\phi = \Gamma v + w$ in (32) and (33). The uniform incident flow field $\mathbf{U}_\infty$ enters the problem data for w as

$$\frac{\partial w}{\partial n} = \mathbf{U}_\infty . \mathbf{n} \quad \text{on } \partial\Omega_f \tag{48}$$

Thus if $\mathbf{U}_\infty$ is changed to correspond to a different angle of attack α, the solution for v remains unchanged and the solution for w apparently requires an additional substitution sweep since the right-side vector in the boundary element system is changed with $\mathbf{U}_\infty$.

In fact one can achieve still greater efficiency by noting that $\mathbf{U}_\infty = |\mathbf{U}_\infty| (\cos\alpha \mathbf{e}_1 + \sin\alpha \mathbf{e}_2)$ where $\mathbf{e}_1$, $\mathbf{e}_2$ are unit base vectors. This implies that the term on the right in (35) can be split to

$$\int_{\partial\Omega_f} \chi \mathbf{U}_\infty . \mathbf{n}\, ds = |\mathbf{U}_\infty| \left(\cos\alpha \int_{\partial\Omega_f} \chi n_1\, ds + \sin\alpha \int_{\partial\Omega_f} \chi n_2\, ds \right) \tag{49}$$

where n_1 and n_2 are the components of the unit outward normal to $\partial\Omega_f$. Let us denote them by w_c and w_s, the parts of w corresponding to incident flows $|\mathbf{U}_\infty|\, \mathbf{e}_1$ and $|\mathbf{U}_\infty|\, \mathbf{e}_2$, respectively; that is, the two flow fields corresponding to $\alpha = 0$ and $\alpha = \pi/2$ in (49). Then, for given α, $w = w_c \cos\alpha + w_s \sin\alpha$ and the potential in (25) becomes:

$$\phi = \Gamma_\alpha v + w_c \cos\alpha + w_s \sin\alpha \tag{50}$$

where Γ_α indicates the dependence of Γ on incident angle α.

Thus, we can solve separate boundary integral equations for v, w_c and w_s and reconstruct ϕ according to (50). In fact, w_c and w_s are obtained as the solution of (35) and hence (41) for incident flows with velocity $|\mathbf{U}_\infty|\, \mathbf{e}_1$ and $|\mathbf{U}_\infty|\, \mathbf{e}_2$. Since the differences in the boundary element systems arise only in the right-side vectors we can again use the factored matrices $\mathbf{L}$ and $\mathbf{U}$, repetitively and solution for w_c and w_s simplifies to two successive forward and backward substitution sweeps. The circulation Γ is again obtained from a relation corresponding to (46). Finally, having obtained v, w_c and w_s we can directly determine the flow field for any angle of incidence α by means of (50) and the Kutta condition for Γ.

NUMERICAL RESULTS

The foregoing boundary element method for lifting aerofoil calculations is now examined in a series of numerical studies for an NACA0012 aerofoil with remote uniform incident flow $\mathbf{U}_\infty$ at a prescribed angle α to the chord of the aerofoil. A non-uniform graded mesh of linear elements with 117 nodes on the aerofoil is employed. The discretization for the aerofoil is indicated in Figure 2 with an expanded detail of the mesh system at the leading edge in Figure 3.

In Figure 4 the pressure coefficient $\mathscr{C}_p$ on the upper and lower surfaces of the aerofoil is plotted for incident flows at angles $\alpha = 0°$, 4°, 6° and 8°. The circulation Γ is calculated using (46) and lift coefficient from (47). Lift coefficient $\mathscr{C}_l$ is plotted against angle of attack α from $\alpha = 0°$ to $\alpha = 20°$ and compared with experimental results[12] in Figure 5. Good agreement is indicated at low and moderate angle of attack. Beyond $\alpha = 14°$ the flow is strongly separated so that the inviscid potential flow assumptions in the mathematical model are no longer valid and the results differ.

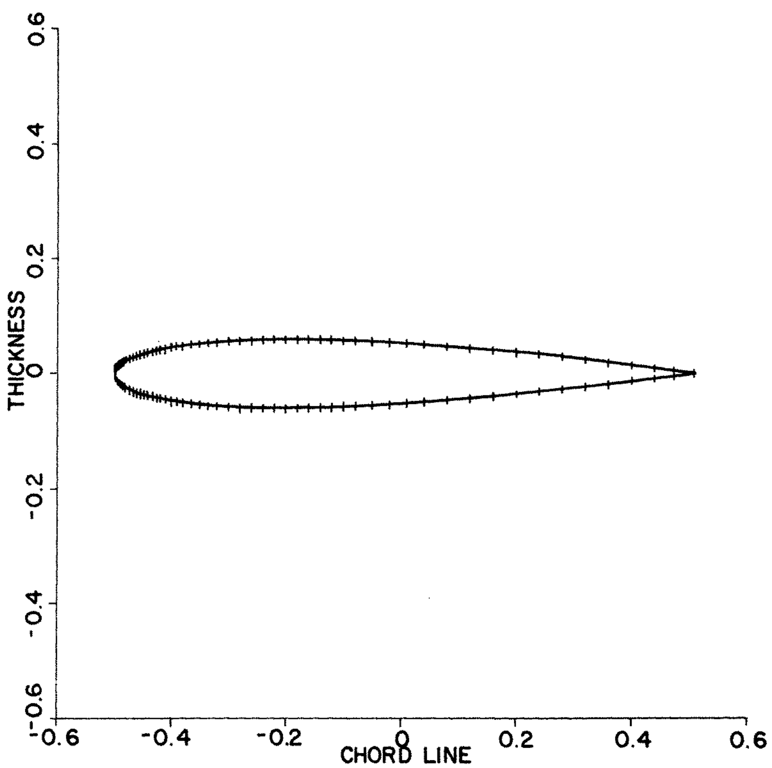

Figure 2. Discretization of linear boundary elements approximating NACA 0012 aerofoil boundary $\partial\Omega_a$

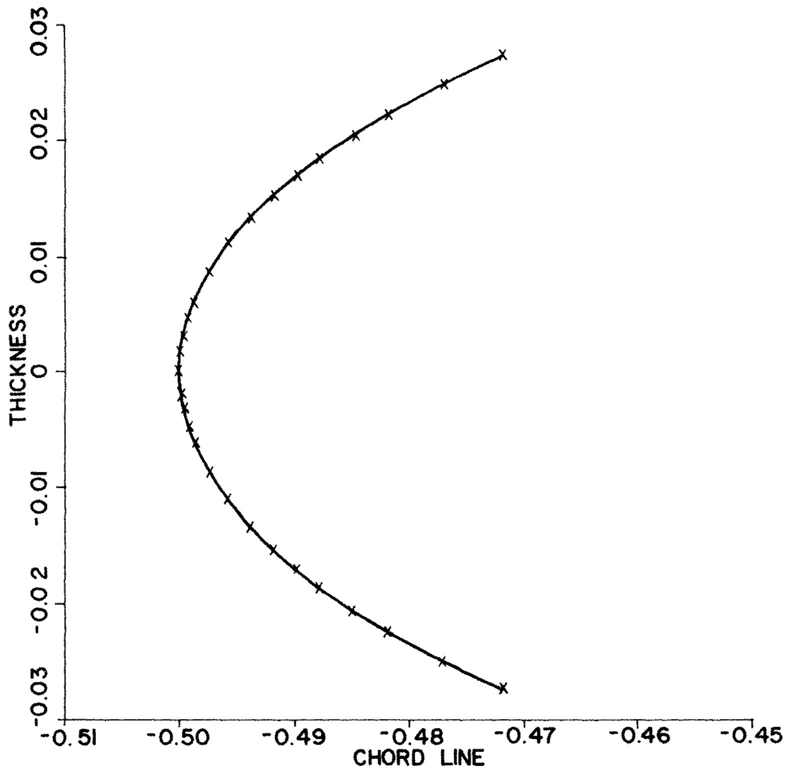

Figure 3. Detail of graded boundary element mesh at leading edge

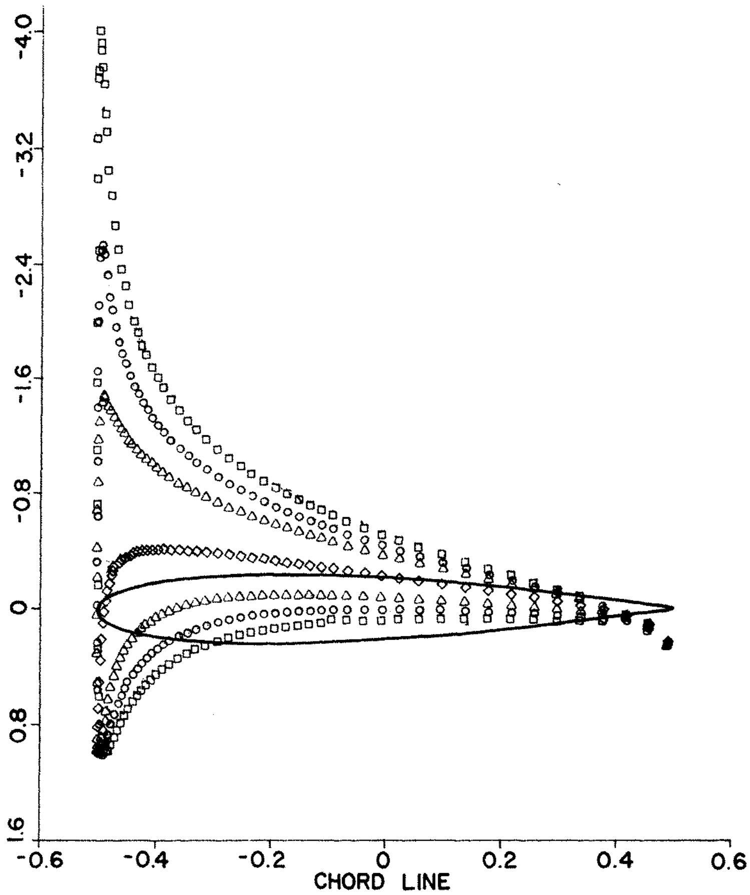

Figure 4. Plots of pressure coefficient $\mathscr{C}_p$ with respect to chord length for incident uniform flow at angles $\alpha = 0°$, 4°, 6°, and 8° (◇, △, ○, □)

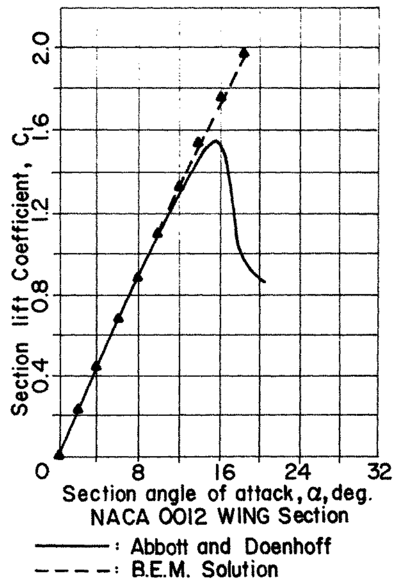

Figure 5. Comparison of computed lift coefficient $\mathscr{C}_l$ with experimental results

CONCLUDING REMARKS

In this study, we have developed an extension of the boundary element method for incompressible potential flows to the case of flows with circulation, and presented numerical results for a lifting aerofoil. Alternative techniques for determining the circulation have been briefly described and a superposition strategy used in the computations. In continuing studies the method is being extended further to consider subcritical compressible lifting flows.

The method, although qualitatively similar to well known panel methods is conceptually distinct in that it is not directly based on a source, sink or vortex superposition principle. In some instances, one may show equivalence between the boundary element method and panel method but this is not true in general.

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